COMBINATORICA
Akadémiai Kiadó — Springer-Verlag

COMBINATORICA 8 (4) 1988 313—321

CONTRACTIONS OF GRAPHS WITH NO SPANNING EULERIAN SUBGRAPHS

P. A. CATLIN

Received March 10, 1986
Revised February 9, 1987

Let $p \geqq 2$ be a fixed integer, and let G be a connected graph on n vertices. If $\delta(G) \geqq 2$, if $d(u)+d(v) > 2n/p-2$ holds whenever $uv \notin E(G)$, and if n is sufficiently large compared to p, then either G has a spanning eulerian subgraph, or G is contractible to a graph G_1 of order less then p and with no spanning eulerian subgraph. The case $p=2$ was proved by Lesniak—Foster and Williamson. The case $p=5$ was conjectured by Benhocine, Clark, Köhler, and Veldman, when they proved virtually the case $p=3$. The inequality is best-possible.

1. Introduction

Consider a finite graph G with vertex set $V(G)$ and edge set $E(G)$. Let n denote the order of G, and let G^c denote the complement of G. Let $d(v)$ denote the degree of v in G, and let $d_1(v)$ denote the degree of v in G_1. The edge-connectivity of G is $\varkappa'(G)$. Let $a(G)$ denote the arboricity of G: i.e., the minimum number of forests whose union contains $E(G)$. We regard eulerian graphs as being connected, and a *spanning eulerian subgraph* of G is an eulerian subgraph containing every vertex of G.

For $xy \in E(G)$, an *elementary contraction* of G is the graph G/xy obtained from G by deleting $\{x, y\}$ and inserting a new vertex v and edges joining v to each $w \in V(G-\{x, y\})$ with as many edges as $\{x, y\}$ was joined to w by edges in G. (Thus, an elementary contraction can create multiple edges). A *contraction* of G is a graph G/H obtained from G by a sequence of elementary contractions of edges of the subgraph H.

Lesniak—Foster and Williamson [6] proved:

Theorem 1. *Let G be a graph of order $n \geqq 6$. If $\delta(G) \geqq 2$ and if any pair u, v of non-adjacent vertices of G,*

$$d(u)+d(v) \geqq n-1, \tag{1}$$

then G has a spanning eulerian subgraph.

Benhocine, Clark, Köhler, and Veldman [1] recently proved:

Theorem 2. *Let G be a 2-edge-connected graph on $n \geqq 3$ vertices. If*

$$d(u)+d(v) \geqq \frac{1}{3}(2n+3)$$

whenever $uv \notin E(G)$, then G has a spanning eulerain subgraph.

AMS subject classification: 05 C 45.

In this paper, we shall generalize these results, using a new method. We first present a concept that we introduced in [2].

A graph G is called *collapsible* if for any even set $S \subseteq V(G)$, there is a forest Γ in G such that both

i) $G-E(\Gamma)$ is connected; and

ii) S is the set of vertices of odd degree in Γ.

We state some observations about collapsible graphs:

(2) The cycles C_3 and C_2 are collapsible.

Note that if G is not 2-edge-connected, or if $G=C_k$ for $k \geqq 4$, then G is not collapsible. Also, $K_{2,t}$ is never collapsible, for any t. If $t \geqq 1$, then K_t is collapsible, except when $t=2$.

(3) If H has two edge-disjoint spanning trees, then H is collapsible.

Statement (3) follows from the fact that for any even subset R of the vertices of a tree T, there is a forest Γ in T such that R is the set of vertices of odd degree in Γ.

What makes the concept of collapsible graphs useful in the study of spanning eulerian subgraphs is the following proposition: Let H be a connected subgraph of G. If H is collapsible, then these are equivalent:

i) G has a spanning eulerian subgraph;

ii) G/H has a spanning eulerian subgraph.

Also, if H is collapsible, then G is collapsible iff G/H is collapsible.

2. The main results

We prove our main result in terms of collapsible graphs, and in the corollaries we express it in terms of spanning eulerian subgraphs.

Theorem 3. *Let G be a connected simple graph of order n, and let $p \geqq 2$ be an integer. If*

$$d(u)+d(v) > \frac{2n}{p}-2 \tag{4}$$

whenever $uv \notin E(G)$, and if

$$n \geqq 4p^2, \tag{5}$$

then exactly one of the following conclusions holds:

a) *G is collapsible;*

b) *G is contractible to a noncollapsible graph G_1 of arboricity $a(G_1) \leqq 2$ and of order less than p;*

c) *$p=2$ and $G-x=K_{n-1}$ for some $x \in V(G)$ with $d(x)=1$;*

d) *$p=4$, and there is a contraction-mapping $G \to C_4$, such that the preimages of some adjacent pair of vertices of C_4 are adjacent singletons of degree 2 in G.*

Also, in every contraction of parts b) *and* d), *the preimage of any vertex of G_1 is an induced collapsible subgraph of G.*

First, we state some consequences of Theorem 3. We regard K_1 as having a spanning eulerian subgraph.

Corollary 1. *Let G be a connected simple graph of order n, and let* $p \geqq 2$ *be an integer. If*

$$d(u)+d(v) > \frac{2n}{p}-2 \tag{6}$$

whenever $uv \notin E(G)$, *and if*

$$n \geqq 4p^2, \tag{7}$$

then exactly one of the following conclusions holds:

a) *G has a spanning eulerian subgraph;*
b) *G is contractible to a graph* G_1 *of order less than p and containing no spanning eulerain subgraph;*
c) $p=2$, *and* $G-x=K_{n-1}$ *for some* $x \in V(G)$ *with* $d(x)=1$.

Corollary 2. *Let G be a 2-edge-connected simple graph of order n. If* $n \geqq 100$ *and if*

$$d(u)+d(v) > \frac{2n}{5}-2 \tag{8}$$

whenever $uv \notin E(G)$, *then* $L(G)$, *the line graph of G, is hamiltonian, and G has a spanning eulerian subgraph.*

Corollary 2 (which is the case $p=5$ of Corollary 1) is a conjecture of Benhocine, Clark, Köhler, and Veldman [1]. The case $p=2$ of Corollary 1 is Theorem 1 (except for the bound on n), due to Lesniak—Foster and Williamson [6]. The case $p=3$ of Corollary 1 is related to Theorem 2, which is a result of Benhocine, Clark, Köhler, and Veldman [1]. In Theorem 7 of [2], we proved a result related to the cases $p=4$ of Theorem 3 and $p=5$ of Corollary 1.

Proof of Corollary 1. Clearly, a) of Theorem 3 implies a) of Corollary 1. The same is true of c). Suppose, in b) and d) of Theorem 3, that the image G_1 of the contraction-mapping $G \to G_1$ has a spanning eulerian subgraph. Since the preimage of each vertex of G_1 is collapsible, it follows easily from the definition of collapsible graphs that G has a spanning eulerian subgraph. If, in b) of Theorem 1, the contraction G_1 has no spanning eulerian subgraph, then neither does G. ∎

Proof of Corollary 2. Set $p=5$ in Corollary 1. Harary and Nash—Williams [4] showed that G has a closed trail containing at least one end of each edge of G iff $L(G)$ is hamiltonian. ∎

Theorem 3 also can be applied to show that G has a spanning (x, y)-trail, for every choice of $x, y \in V(G)$. For this conclusion to hold, the hypothesis of Corollary 2 is not sufficient when G satisfies d) of Theorem 3. It would suffice if (8) is replaced by

$$d(u)+d(v) > \frac{n}{2}.$$

This is best possible.

Corollary 3. *Let G be a 3-edge-connected simple graph of order n. If n is sufficiently large and if*

$$d(u)+d(v) > \frac{n}{5}-2 \tag{9}$$

whenever $uv \notin E(G)$, *then G has a spanning eulerian subgraph.*

Proof. Set $p=10$ in Corollary 1. If a) fails, then b) holds. By the definition of contractions,

$$\varkappa'(G_1) \geqq \varkappa'(G),$$

and so G_1 is 3-edge-connected. By inspection, there is no 3-edge-connected graph of order less than p with no spanning eulerian subgraph. Therefore, b) cannot hold. ▮

Jaeger [5] showed that a graph containing two edge-disjoint spanning trees has a spanning eulerian subgraph (such a graph is also collapsible, by (3)). We have also used this method of collapsible graphs in another paper [3], to obtain other conditions for a graph to have a spanning eulerian subgraph.

Let G_1 be a graph of order p satisfying

i) G_1 has no spanning eulerian subgraph; and

ii) Any contraction G' of G_1 has a spanning eulerian subgraph.

The only such graphs G_1 of order at most 7 are K_2, $K_{2,3}$, $K_{2,5}$, and Q_3-v (the cube minus a vertex).

We claim that for any $p\geqq 7$, there is a graph G_1 of order p satisfying both i) and ii). When p is odd, $G_1=K_{2,\,p-2}$ is such a graph. We shall construct examples for even values of p, next. Let H be a path of length 3 with consecutive vertices labelled x_1, x_2, x_3, x_4. Define the graph $G(s, t)$ of order $4+s+t$, to be the graph obtained from H by adding s vertices with neighbourhood $\{x_1, x_3\}$ and t vertices with neighbourhood $\{x_2, x_4\}$. Suppose s and t are even. Then the set S of odd-degree vertices of $G(s, t)$ is $S=\{x_1, x_4\}$. Because of the set S, $G(s, t)$ is not collapsible, for if Γ is a forest in $G(s, t)$, with S as the set of odd-degree vertices of Γ, then $G(s, t)-E(\Gamma)$ is not connected. Therefore, if s and t are even, then $G(s, t)$ has no spanning eulerian subgraph, and so for any even integer $p\geqq 8$, $G_1=G(2, p-6)$ satisfies condition i) above. Since G_1 also satisfies ii), our claim is true.

We shall now show that the inequalities (4), (6), (8), and (9) are best-possible.

Form the graph G by replacing each vertex of G_1 with a clique K_s $(s\geqq 1)$, such that the edges of $E(G_1)$ join the corresponding cliques in G, and so that G has order $n=ps$ and is contractible to G_1, of order p. Since G_1 has no spanning eulerian subgraph, neither has G, and neither G_1 nor G is collapsible. Whenever $uv\notin E(G)$,

$$d(u)+d(v) \geqq \frac{2n}{p}-2,$$

and if $s>\Delta(G_1)$, then equality holds for some $u, v\in V(G)$. Thus, (4) and (6) barely fail, and the conclusions of Theorem 3 and Corollary 1 fail. When $G_1=K_{2,3}$, the corresponding G shows that (8) of Corollary 2 is best-possible, and when G_1 is the Petersen graph, the corresponding G shows that (9) of Corollary 3 is best-possible.

Corollary 3 holds even when its conclusion is changed to "G is collapsible".

With a longer argument, it is possible to improve (5) and (7) to $n\geqq p^2$, except for the following cases:

$$p = 2,\ n = 5,\ G = G_1 = K_{2,3};$$

$$p = 5,\ n \leqq 32,\ G_1 = K_{2,3};$$

$p = 6,\ n \leqq 38,\ G_1$ is the bipartite theta graph of order 6.

The first exceptional case arises in Theorem 1. In the latter two exceptional cases, as in d) of Theorem 3, there are two adjacent vertices $x, y \in V(G_1)$, such that $d_1(x)+d_1(y)=p$ and the preimages of x and y in G are singletons with $d(x)+d(y)=p$ in G. It also appears possible that even $n \geqq p^2$ is not quite best-possible, but it is close. The details are tedious, and we omit them.

If the proof that follows were a proof of Corollary 1 directly, we would still define G_1 exactly as in the beginning of the proof that follows, in terms of contractions of collapsible subgraphs of G.

3. The proof

The conclusions a), b), c), and d) of Theorem 3 are mutually exclusive.

Let G be a connected simple graph satisfying (4) and (5), but not a) of Theorem 3. Let $E \subseteq E(G)$ be a minimal edge-set such that every component $H_1, H_2, \ldots, H_c$ of $G-E$ is collapsible. Since a) fails, G is not collapsible, and since each component K_1 of $G-E(G)$ is collapsible, E exists. If G has a cut-edge and $p>2$ then let G_1 be a K_2 (note that (b) is satisfied and we are done); but if G has no cut-edge or if $p=2$, then let G_1 denote the graph obtained from G by contracting all edges of $E(G)-E$. Since $\omega(G-E)=c$,

(10) $$c = |V(G_1)|.$$

By the minimality of E, and by (2),

(11) G_1 has no 3-cycle; and

(12) G_1 has no multiple edges.

Since $|E(G_1)| \geqq 2c-2$ implies that some nontrivial induced subgraph H of G_1 contains two edge-disjoint spanning trees, both $a(G_1) \geqq 2$ and

(13) $$|E| = E(G_1)| \leqq 2c-3$$

follow from (3), the minimality of E, and (10). (By results we obtained in [2], either G_1 has a bridge or the inequality (13) is strict.)

If G has a cut-edge, then since G is simple, is straightforward to show that (4) of Theorem 3 implies either conclusion b) ($p>2$; G_1 has a cut-edge) or conclusion c) ($p=2$). Hence, we shall suppose $\varkappa'(G) \geqq 2$ and hence that

(14) $$\varkappa'(G_1) \geqq 2.$$

Since the smallest 2-edge-connected noncollapsible graph is $G_1=C_4$, we may suppose, without loss of generality, that

(15) $$c \geqq 4.$$

We shall use the following lemma:

Lemma. *Let H be a graph, and for each $x \in V(H)$, define*

$$B(x) = \{w \in V(H) | wx \in E(H^c)\}$$

If H is triangle-free, and not a star, then the family $(B(x)|x \in V(H))$ has a complete system of distinct representatives.

Proof. Let H be triangle-free and not a star. If H is the five-cycle, then the lemma holds. We claim that if $H \neq C_5$, then H^c has a spanning subgraph in which each component is either K_2 or K_3. Note that the lemma follows, if we prove this claim.

By way of contradiction, suppose

(16) H is a smallest counterexample to the claim.

By inspection, we may suppose

(17) $$|V(H)| \geqq 6.$$

Since H is triangle-free and since the Ramsey number $r(3, 3)$ is 6, (17) implies that H contains an independent set $\{x, y, z\} \subseteq V(H)$.

If $H-\{x, y, z\}$ is not a star, then by (16), $H-\{x, y, z\}$ satisfies the claim. Since $H^c[\{x, y, z\}]=K_3$, H satisfies the claim, contrary to (16).

If $H-\{x, y, z\}$ is a star, then by (17), $H-\{x, y\}$ is not a star. By (16), $H-\{x, y\}$ satisfies the claim, and since $H^c[\{x, y\}]=K_2$, H satisfies the claim, contrary to (16). ∎

Proof of Theorem 3, continued. Since a) fails for G, a) also fails for G_1 (see [2], Theorem 6). This satisfies a requirement of b). Let

$$V(G_1) = \{x_1, x_2, \ldots, x_c\}.$$

By (11) and (14), G_1 satisfies the hypotheses of the lemma. Therefore, there is a system of distinct representatives $y_i \in B(x_i)$ $(1 \leqq i \leqq c)$. The resulting set of ordered pairs

$$\{(x_1, y_1), (x_2, y_2), \ldots, (x_c y_c)\}$$

corresponds to a set E'' (with multiplicities allowed) of c edges of G_1^c:

$$E'' = \{x_1 y_1, x_2 y_2, \ldots, x_c y_c\}.$$

(A multiplicity occurs when $x_i = y_j$ and $x_j = y_i$ for some i and j.) Let

$$G'' = G_1^c[E'']$$

be the subgraph of G_1^c induced by E''. Clearly,

(18) G'' is a spanning subgraph of G_1^c;

(19) Each component of G'' is either a K_2 or a cycle; and

(20) An edge occurs more than once in E'' iff it occurs exactly twice, and is the edge of a K_2 component of G''.

Let $\Theta: G \to G_1$ denote the contraction-mapping defining G_1. For each edge $xy \in E(G'') \subseteq E(G_1^c)$, the preimages $\Theta^{-1}(x)$ and $\Theta^{-1}(y)$ are distinct components of $G-E$, with no edge of G joining a vertex of $\Theta^{-1}(x)$ to a vertex of $\Theta^{-1}(y)$. For all i with $1 \leqq i \leqq c$, pick $u_i \in \Theta^{-1}(x_i)$ and $v_i \in \Theta^{-1}(y_i)$. Then $u_i v_i \in E(G^c)$, for all i. Denote by E' the set (possible with multiplicities)

$$E' = \{u_1 v_1, u_2 v_2, \ldots, u_c v_c\}.$$

Hence, $\Theta[E'] = E''$, and by (18), each component of $G-E$ contains one member of $U = \{u_1, u_2, \ldots, u_c\}$; and since $\{y_1, y_2, \ldots, y_c\}$ is a transversal, each component of $G-E$ contains one member of $V = \{v_1, v_2, \ldots, v_c\}$.

Define $N_{G-E}(x)$ to be the neighbourhood of x in $G-E$. If for some j and k, $u_i \in V(H_j)$ and $v_i \in V(H_k)$, then

$$|N_{G-E}(u_i)|+|N_{G-E}(v_i)| \leqq |V(H_j)|-1+|V(H_k)|-1. \tag{21}$$

Since each component of $G-E$ contains exactly one $u_i \in U$ and one $v_i \in V$, we can sum (21) over E' and get

$$\sum_{i=1}^{c} (|N_{G-E}(u_i)|+|N_{G-E}(v_i)|) \leqq$$

$$\leqq \sum_{j=1}^{c} |V(H_j)|-1+\sum_{k=1}^{c} |V(H_k)|-1 = 2(n-c).$$

By $E=E(G_1)$ and by (13), there are at most $2|E| \leqq 4|V(G_1)|-6=4c-6$ incidences in G of edges of E with U and at most $2|E| \leqq 4c-6$ incidences in G of edges of E with V. Hence,

$$\sum_{i=1}^{c} d(u_i)+d(v_i) = 4|E|+\sum_{i=1}^{c} (|N_{G-E}(u_i)|+|N_{G-E}(v_i|) \leqq \tag{22}$$

$$\leqq 2(4c-6)+2(n-c) = 2n+6c-12.$$

Finally, we are ready to use the hypothesis of Theorem 3. Since $u_i v_i \in E(G^c)$, (4) and (22) give

$$c\left(\frac{2n}{p}-2\right) < \sum_{i=1}^{c} d(u_i)+d(v_i) \leqq 2n+6c-12$$

$$c(n-4p) < np-6p$$

$$c < \frac{np-6p}{n-4p} \tag{23}$$

which is less than $p+1$, by (5). Hence, by (10),

$$|V(G_1)| = c \leqq p \tag{24}$$

Suppose that (24) holds with equality; i.e., suppose

$$c = p \tag{25}$$

We then show that we have case d).

Arrange the components $H_1, H_2, \ldots, H_c$ of $G-E$ such that

$$|V(H_1)| \leqq |V(H_2)| \leqq \ldots \leqq |V(H_c)|. \tag{26}$$

Case I. Suppose $|V(H_1)| > \Delta(G_1)$.

Then in (22) we can choose u_i and v_i, for $1 \leqq i \leqq c$, so that they are not incident with E, and so the $4|E|$ term disappears from (22):

$$\sum_{i=1}^{c} d(u_i)+d(v_i) \leqq 2(n-c). \tag{27}$$

By (4) and (27),

$$c\left(\frac{2n}{c}-2\right) < 2(n-c), \tag{28}$$

a contradiction. Therefore,

$$|V(H_1)| \leqq \Delta(G_1) \leqq c-1, \tag{29}$$

and the latter inequality follows from (12).

Case II. Suppose that for every $i \geqq 2$, there is an $x_1 \in V(H_1)$ and an $x_i \in V(H_i)$ such that $x_1 x_i \notin E(G)$. By (4), (25), (29), and (13), for $i \geqq 2$,

$$\frac{2n}{c}-2 < d(x_1)+d(x_i) \leqq \tag{30}$$

$$\leqq E|+|V(H_1)|+|V(H_i)|-2 \leqq 3c-4+|V(H_i)|-2.$$

We sum (30) over all $i \geqq 2$ to get

$$(c-1)\frac{2n}{c} < (3c-4)(c-1)+\sum_{i=1}^{c}|V(H_i)| <$$

$$< (3c-4)(c-1)+n,$$

which, by (15), is false for large n.

Case III. Suppose that for some $k \geqq 2$, $x_1 x_k \in E(G)$ for all $x_1 \in V(H_1)$ and $x_k \in V(H_k)$. Since G_1 is simple, by (12), this implies

$$|V(H_1)| = |V(H_k)| = 1. \tag{31}$$

Suppose $k \geqq 3$. Then (31) and (26) imply $V(H_i) = \{x_i\}$ for $1 \leqq i \leqq 3$. By (11), two of $\{x_1, x_2, x_3\}$ are not adjacent in G, say x_1 and x_2. Then by (4), (25), and (13),

$$\frac{2n}{c}-2 < d(x_1)+d(x_2) \leqq |E| \leqq 2c-3,$$

which contradicts (5). Hence, $k=2$ and

$$|V(H_j)| > 1, \tag{32}$$

if $j \geqq 3$. By (12),

$$d(x_1) \leqq c-1, \tag{33}$$

and at most one edge of $E(G)$ joins $V(H_1)$ and $V(H_j)$ $(3 \leqq j \leqq c)$. Thus by (32) there is an $x_j \in V(H_j)-N(x_1)$ whenever $3 \leqq j \leqq c$, and so

$$\sum_{j=3}^{c} d(x_j) \leqq (2|E|-1)+\sum_{j=1}^{c}\left(|V(H_j)|-1\right) \leqq (2|E|-1)+n-c. \tag{34}$$

By (4), (13), (25), (33), and (34),

$$(c-2)\frac{2n}{c} < \sum_{j=3}^{c} d(x_1)+d(x_j) \leqq (c-2)\,d(x_1)+(n-c+2|E|-1) \leqq \tag{35}$$

$$\leqq (c-2)(c-1)+n+3c-7.$$

By (15), $c \geqq 4$. Unless $c=4$, (35), (5), and (15) combine to give a contradiction. When $c=4$, (25) and (31) and $k=2$ imply that d) of Theorem 3 holds.

This completes Case III and the proof of Theorem 3. ∎

References

[1] A. Benhocine, L. Clark, N. Köhler and H. J. Veldman, On circuits and pancyclic line graphs, *Preprint*, 1985.
[2] P. A. Catlin, Removing forests to find spanning eulerian subgraphs, *Preprint*, 1985.
[3] P. A. Catlin, Spanning eulerian subgraphs and matchings, *Preprint*.
[4] F. Harary and C. St. J. A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs, *Canadian Math. Bull* **8** (1965), 701—710.
[5] F. Jaeger, A note on subeulerian graphs, *J. Graph Theory* **3** (1979), 91—93.
[6] L. Lesniak-Foster and J. E. Williamson, On spanning and dominating circuits in graphs, *Canad. Math. Bull.* **20** (1977), 215—220.

Paul A. Catlin

Wayne State University
Detroit, Michigan 48202, U.S.A.